ANALYTIC SKEW PRODUCTS OF QUADRATIC POLYNOMIALS OVER CIRCLE EXPANDING MAPS

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ABSTRACT. We prove that a Viana map with an arbitrarily non-constant real analytic coupling function admits two positive Lyapunov exponents almost everywhere.

1. INTRODUCTION

We study skew products of quadratic maps driven by expanding circle maps with real-analytic coupling functions. Let $f(x) = a - x^2$, $a \in (1, 2)$ be a quadratic map for which $x = 0$ is a strictly pre-periodic point, let $d \geq 2$ be an integer and let $\phi : \mathbb{T} \to \mathbb{R}$ be a non-constant real-analytic function, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For $\alpha > 0$, define $F = F_\alpha : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ as

$$F(\theta, x) = (d\theta, f(x) + \alpha\phi(\theta)).$$

These kind of maps, nowadays called *Viana maps*, were first studied in [9], where Viana proved that for $d \geq 16$, $\phi(\theta) = \sin(2\pi\theta)$, if α is small enough, then F_α has two positive Lyapunov exponents Lebesgue almost everywhere. Furthermore, he showed that the same conclusion holds for any small C^3 perturbation of F. In [5], Buzzi-Sester-Tsujii extended the result to the case $d \geq 2$ (for the same coupling function ϕ). The non-integer case of d was considered in [7]. See also [8, 6] for results on skew products driven by certain quadratic polynomials. These maps serve as typical examples of multi-dimensional non-uniformly expanding dynamical systems for which a general theory have been developed, see the excellent review [2].

An important feature of the maps F considered in the papers cited above is that they display partial hyperbolicity: the maps are uniformly expanding in the horizontal direction and the horizontal expansion dominates the vertical expansion. The property implies that images of horizontal curves are nearly horizontal. The particular form of the coupling function ϕ allows the authors to conclude that high iterates of a horizontal curve are non-flat, which is an important technical point in the proof.

The goal of this paper is to weaken the assumption on the coupling function.

Theorem 1. *Fix f, g, ϕ as above. If α is small enough, then F has two positive Lyapunov exponents at Lebesgue almost every point in $\mathbb{T} \times \mathbb{R}$. Moreover, the same holds for any small C^∞ perturbation of F.*

We shall only deal with the unperturbed case, as the perturbed case follows by the same strategy in [9]. As in the previous works, we analyze return of an "*admissible curve*" (or *a sufficiently high iterate of a horizontal curve*) to small neighborhoods of the critical circle $x = 0$. The difference here from [5] is that for general ϕ, the images of an admissible curve are not necessarily separated from each other on a definite part, so that a key estimate (Lemma 2.6 in [9] and Proposition 5.2 in [5]) only holds in a weaker form. Nevertheless, we shall prove that most points in an admissible curve cannot return too close to the critical

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circle before displaying some (weak) expansion, see Proposition 3. The real-analyticity of the coupling function is essentially used in § 3.1 to obtain non-flatness of admissible curves. It is not clear to us whether the same result holds for smooth coupling functions ϕ with non-flat critical points.

Let us mention a few consequences of our theorem. Since we do not obtain exactly the same underlying estimates as in [9], the proof in [1] does not apply directly in our case to show existence of SRB measures. However, our result shows that F (and any small C^∞ perturbation) satisfies the assumption of [3], from which we conclude that F admits finitely many acip's. Moreover, a slight modification of [4] shows that F is ergodic with respect to the Lebesgue measure. Thus, in our case, F and its small C^∞ perturbation admit a unique acip.

Note. Let $f_n : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be defined so that

$$F^n(\theta, x) = (d^n\theta, f_n(\theta, x)).$$

Choose β slightly smaller than $|p_1|$, where $p_1 = \frac{-1-\sqrt{1+4a}}{2}$ is the orientation-preserving fixed point of f. Then $1 < \beta < 2$ and the interval $B = [-\beta, \beta]$ satisfies: $f(B) \subset \text{int}(B)$ and $|f_a'(x)| \geq 2\beta$ for $x \in \mathbb{R} \setminus \text{int}(B)$. Let $I = \mathbb{T} \times B$. Then provided that $\alpha > 0$ is small enough, the following hold:

- $F(I) \subset \text{int}(I)$.
- $|\frac{\partial f_1(\theta,x)}{\partial x}| > \beta > 1$ outside of I.
- $|\frac{\partial f_1(\theta,x)}{\partial x}| \leq 2\beta \leq 4$ on I.

Note. Unless otherwise stated, all constants appearing below depend on f, d and ϕ. Dependence of constants on α will be stated explicitly.

Without loss of generality, we assume $|\phi(\theta)| \leq 1$ for all $\theta \in \mathbb{T}$.

2. Preliminaries

2.1. **Domination.** An important feature of the map F is that the horizontal expansion dominates the vertical expansion.

Lemma 1. *There exists $C_1 > 0$, $R_1 \in (0, 2)$ and $n_0 \in \mathbb{N}$ such that, when α is sufficiently small, for all $(\theta, x) \in I$, we have*

$$\left|\frac{\partial f_k(\theta, x)}{\partial x}\right| \leq C_1 R_1^k \text{ for } k \in \mathbb{N} \text{ and } \left|\frac{\partial f_k(\theta, x)}{\partial x}\right| \leq R_1^k \leq \frac{d^k}{6} \text{ for } k \geq n_0.$$

Proof. Following from the proof of Lemma 3.1 in [5], one can show that there exists $0 < C_1 < \infty$ and $R_* \in (0, 2)$ such that when α is sufficiently small, for all $(\theta, x) \in I$, we have

$$\left|\frac{\partial f_k(\theta, x)}{\partial x}\right| \leq C_1 R_*^k \text{ for } k \in \mathbb{N}.$$

Then we take $R_1 \in (R_*, 2)$ and $n_0 \in \mathbb{N}$ satisfying $C_1 R_*^{n_0} \leq R_1^{n_0} \leq \frac{d^{n_0}}{6}$. This gives us the desired estimate. ☐

2.2. **Building expansion.** The following is a variation of Lemma 2.5 in [9]. The same proof works.

Given $(\theta, x) \in \mathbb{T} \times B$, write $(\theta_j, x_j) = F^j(\theta, x)$.

Lemma 2. *There are $\lambda > 1$, $K > 0$ and $\delta > 0$ such that, when α is sufficiently small,*

$$\left|\frac{\partial f_n(\theta, x)}{\partial x}\right| \geq K\alpha^{0.6}\lambda^n$$

for all $(\theta, x) \in \mathbb{T} \times B$ with $|x_0|, |x_1|, \ldots, |x_{n-1}| \geq \alpha^{0.6}$. If, in addition, $|x_n| < \delta$ then we even have

$$\left|\frac{\partial f_n(\theta, x)}{\partial x}\right| \geq K\lambda^n.$$

2.3. **Shadowing.** We present a more precise version of Lemma 2.4 in [9] here.

Fix a constant $\xi_0 > 0$ such that for all $i \geq 1$,

$$c_i := f^i(0) \notin (-2\xi_0, 2\xi_0).$$

Assume $\alpha \in (0, \xi_0]$. Define $I_1 = [c_1 - 2\alpha, c_1 + 2\alpha]$, and for $n \geq 1$, inductively define I_{n+1} be the closed interval with the following property: $I_{n+1} \Supset f(I_n)$ and both components of $I_{n+1} \setminus f(I_n)$ have length α. Let $N(\alpha)$ be the maximal integer such that the following hold:

- $|I_n| < \xi_0$ for all $n = 1, 2, \ldots, N(\alpha)$;
- $\sum_{n=1}^{N(\alpha)} |I_n| \leq \xi_0$.

Note that $|\phi| \leq 1$. Then $F : \mathbb{T} \times I_n \to \mathbb{T} \times I_{n+1}$ is fibre-wise diffeomorphic, for any $n = 1, 2, \ldots, N(\alpha)$.

Lemma 3. *For all $1 \leq m < n \leq N(\alpha) + 1$, and for any $x_i \in I_i$, $i = m, m+1, \ldots, n-1$, we have*

$$e^{-1}|(f^{n-m})'(c_m)| \leq \prod_{i=m}^{n-1} |f'(x_i)| \leq e|(f^{n-m})'(c_m)|.$$

Proof. In fact, $|c_i - x_i| \leq |I_i|$ for all $m \leq i < n$. Thus

$$\left|\log \frac{\prod_{i=m}^{n-1} |f'(x_i)|}{|(f^{n-m})'(c_m)|}\right| = \sum_{i=m}^{n-1} \log\left|1 + \frac{x_i - c_i}{c_i}\right| \leq \sum_{i=m}^{n-1} \frac{|x_i - c_i|}{|c_i|}$$
$$\leq \xi_0^{-1} \sum_{i=m}^{n-1} |I_i| \leq 1.$$

The statement follows. □

Lemma 4. *There exists $C_0 \in (0, 1)$ such that*

$$C_0 \leq |(f^{N(\alpha)})'(c_1)|\alpha \leq \frac{1}{C_0}.$$

Proof. By the mean value theorem, for each n, there exists $x_n \in I_n$ such that

$$|I_{n+1}| = D_n|I_n| + 2\alpha,$$

where $D_n = |f'(x_n)|$. It follows that for all $0 \le n \le N(\alpha)$,

$$\begin{aligned}|I_{n+1}| &= D_n|I_n| + 2\alpha \\ &= D_n D_{n-1}|I_{n-1}| + 2\alpha(1+D_n) \\ &= \cdots \\ &= D_n D_{n-1}\dots D_1|I_1| + 2\alpha(1 + D_n + D_n D_{n-1} + \dots + D_n D_{n-1}\dots D_2) \\ &= 2\prod_{i=1}^{n} D_i \left(2\alpha + \alpha \sum_{k=1}^{n}\prod_{i=1}^{k} D_i^{-1}\right).\end{aligned}$$

By Lemma 3, $\prod_{i=1}^{k} D_i^{-1} \asymp |(f^k)'(c_1)|^{-1}$ is exponentially small in k. So

$$|I_{n+1}| \asymp |(f^n)'(c_1)|\alpha.$$

Since $\sum_{n=1}^{N(\alpha)} |(f^n)'(c_1)|/|(f^{N(\alpha)})'(c_1)|$ is bounded from above by a constant depending only on f, the conclusion follows by the definition of $N(\alpha)$. $\square$

Lemma 5. *The following holds provided that α is small enough: for each (θ, x) with $|x| < \sqrt{\alpha}$,*

$$\left|\frac{\partial f_{N(\alpha)}(f_1(\theta,x))}{\partial x}\right| \ge \frac{C_0}{\alpha},$$

where $C_0 > 0$ is a universal constant.

Proof. Note that $f_1(\theta, x) \in I_1$. Thus there exists $x_i \in I_i$, $i = 1, 2, \dots, N(\alpha)$ such that

$$\left|\frac{\partial f_{N(\alpha)}(f_1(\theta,x))}{\partial x}\right| = \prod_{i=1}^{N(\alpha)} |f'(x_i)|.$$

Applying Lemmas 3 and 4 gives us the result by redefining C_0. $\square$

3. Admissible curves

We prove that the images of horizontal curves under a sufficiently high iterate of F, which will be called *admissible curves*, are nearly horizontal and non-flat.

3.1. A class of functions.

We introduce a special class of functions which will be needed in the argument below.

Let $\hat{\phi}(\theta) = \phi(\theta \mod 1)$. Then $\hat{\phi}$ is a real-analytic map from $\mathbb{R}$ to $\mathbb{R}$ with period 1: $\hat{\phi}(\theta+1) = \hat{\phi}(\theta)$. We say that a function $T : \mathbb{R} \to \mathbb{R}$ is in *the class* $\mathcal{T}_\phi$ if there exist $a_n \in \mathbb{R}$, and $k_n \in \{0, 1, \dots, d^n - 1\}$, $n = 1, 2, \dots$ such that the following hold:

- $T(\theta) = \hat{\phi}'(\frac{\theta+k_1}{d}) + \sum_{n=1}^{\infty} a_n \hat{\phi}'(\frac{\theta+k_n}{d^{n+1}})$;
- $|a_n| \le C_1(\frac{R_1}{d})^n$,

where C_1 and R_1 are as in Lemma 1. Take $\rho > 0$ such that $\hat{\phi}$ extends to a holomorphic function defined in $S = \mathbb{R} \times (-2\rho, 2\rho)$. Then each $T \in \mathcal{T}_\phi$ may be viewed as holomorphic mappings defined on S. Clearly, $\mathcal{T}_\phi$ is a compact family with respect to the topology defined by locally uniformly convergence in S.

Lemma 6. *There exists $l_0 \geq 2$, $A > \mu > 0$ depending only on ϕ such that for each $T \in \mathcal{T}_\phi$, and any $\theta \in [0,1]$,*

$$\frac{Ad}{2} \geq \sum_{i=0}^{l_0+1} |T^{(i)}(\theta)| \geq \sum_{i=0}^{l_0} |T^{(i)}(\theta)| \geq 2d\mu. \tag{1}$$

Moreover,

$$\sup_{\theta \in [0,1]} |T(\theta)| \geq 2\mu. \tag{2}$$

Proof. By compactness of the family $\mathcal{T}_\phi$, it suffices to show that $0 \notin \mathcal{T}_\phi$. To this end, we first observe that for any $l \in \mathbb{N}$, $M_l = \sup |\hat{\phi}^{(l)}| > 0$. Choose l sufficiently large such that

$$C_1 \frac{R_1 d^{-l}}{1 - R_1 d^{-l}} \leq 1/2.$$

Now take $T \in \mathcal{T}_\phi$. Choose $\theta_0 \in [0,1)$ such that $|\hat{\phi}^{(l)}(\theta_0)| = M_l$, and we have

$$\begin{aligned} d^{l-1} \left|T^{(l-1)}(d\theta_0 - k_1)\right| &\geq M_l - M_l C_1 \sum_{n=1}^{\infty} \left(\frac{R_1}{d^l}\right)^n \\ &\geq M_l \left(1 - C_1 \frac{R_1 d^{-l}}{1 - R_1 d^{-l}}\right) \\ &\geq \frac{1}{2} M_l. \end{aligned}$$

This shows that T is not identically zero, completing the proof. □

3.2. **Partial derivatives of F.** Now let d, ϕ, α, F be as in Theorem 1, and let $g(\theta) = d\theta$ mod 1. Recall that f_n is the function for which

$$F^n(\theta, x) = (g^n(\theta), f_n(\theta, x)).$$

Write

$$H_n(\theta, x) = \frac{\partial f_n(\theta, x)}{\partial \theta} \text{ and } V_n(\theta, x) = \frac{\partial f_n(\theta, x)}{\partial x}.$$

Then

$$V_n(\theta, x) = (f^n)'(x) + \sum_{m=1}^{2^{n-1}} \alpha^m P_{m,n}(\theta, x), \tag{3}$$

$$H_n(\theta, x) = \alpha d^{n-1} G_n(\theta, x) + \sum_{2 \leq m \leq 2^{n-1}} \alpha^m Q_{m,n}(\theta, x), \tag{4}$$

where $P_{m,n}, Q_{m,n}$ are real analytic functions defined on $\mathbb{T} \times \mathbb{R}$, and

$$G_n(\theta, x) = \sum_{k=1}^{n} \frac{(f^{n-k})'(f^k x)}{d^{n-k}} \phi'(g^{k-1}\theta). \tag{5}$$

We identify $\mathbb{T}$ with $[0,1)$ and let $\mathcal{P}_n$ be the partition of $\mathbb{T}$ consisting of intervals $[j/d^n, (j+1)/d^n)$, $j = 0, 1, \ldots, d^n - 1$.

Remark. For each $\omega \in \mathcal{P}_n$ and $x \in B$, it is easily seen that there exists $T_{\omega,x} \in \mathcal{T}_\phi$ such that

$$G_n(\theta, x) = T_{\omega,x}(g^n\theta) \text{ for all } \theta \in \omega.$$

For any curve $X : \omega \to \mathbb{R}$, $\omega \in \mathcal{P}_n$, let $F^n(X|\omega)$ denote the curve $Y : [0,1) \to \mathbb{R}$ with

$$Y(\theta) = f_n(\tau(\theta), X(\tau(\theta))),$$

where $\tau = (g^n|_\omega)^{-1}$.

Let $\mathcal{H}_C(\alpha)$ denote the set of smooth curves $X : [0,1) \to B$ for which there exists $T \in \mathcal{T}_\phi$ such that $X' - \alpha d^{-1}T$ has C^{l_0} norm bounded from above by $C\alpha^2$. Let n_0 be the positive integer specified in Lemma 1.

Lemma 7. *There exists a constant $C > 0$ such that the following holds provided that $\alpha > 0$ is small enough. Let X be a curve in the class $\mathcal{H}_{2C}(\alpha)$, $\omega \in \mathcal{P}_n$ with $n \geq n_0$, and let $Y = F^n(X|\omega)$. Then $Y \in \mathcal{H}_C(\alpha)$.*

Proof. Let $C > 0$ be a large constant to be determined.

Assume first that $n_0 \leq n \leq 2n_0$. By assumption on X, there exists $T \in \mathcal{T}_\phi$ such that the C^{l_0}-norm of $X' - \alpha d^{-1}T$ is bounded from above by $2C\alpha^2$. Let

$$S(\theta) = \sum_{k=1}^{n} \frac{(f^{n-k})'(x)}{d^{n-k}} \phi'(g^{k-1} \circ \tau(\theta)),$$

and

$$\hat{T}(\theta) := S(\theta) + \frac{(f^n)'(x)}{d^n} T(\tau\theta),$$

where $\tau = (g^n|\omega)^{-1}$ and $x = X(\tau(0))$. Then $S, \hat{T} \in \mathcal{T}_\phi$ by Lemma 1. Recall that

$$Y(\theta) = f_n(\tau(\theta), X(\tau(\theta))),$$

hence

$$Y'(\theta) = H_n(\tau\theta, X(\tau\theta))\tau'(\theta) + V_n(\tau\theta, X(\tau\theta))X'(\tau(\theta))\tau'(\theta).$$

Let us estimate

$$Y'(\theta) - \alpha d^{-1}\hat{T}(\theta) = Z_1(\theta) + Z_2(\theta) + Z_3(\theta),$$

where

$$Z_1(\theta) := H_n(\tau\theta, X(\tau\theta))\tau'(\theta) - \alpha\frac{1}{d}S(\theta)$$

$$Z_2(\theta) := \frac{V_n(\tau\theta, X(\tau\theta)) - (f^n)'(x)}{d^n} X'(\tau\theta),$$

$$Z_3(\theta) := \frac{(f^n)'(x)}{d^n} \left(X'(\tau\theta) - \alpha d^{-1}T(\tau\theta)\right).$$

By (4), (5) and (3), we see that there exists a constant K_n such that for $i = 1, 2$,

$$\|Z_i\|_{l_0} \leq K_n\alpha \cdot \|X(\tau(\theta)) - x\|_{l_0}.$$

Let

$$C = 3A \max_{n_0 \leq n \leq 2n_0} K_n,$$

where A is as in Lemma 6. Then provided that α is small enough,

$$\|X(\tau(\theta)) - x\|_{l_0} \leq \|X'\|_{l_0} \leq \alpha d^{-1}\|T\|_{l_0} + 2C\alpha^2 \leq \alpha A,$$

hence

$$\|Z_i\|_{l_0} \leq C\alpha^2/3, \quad i = 1, 2.$$

By choice of n_0, we have

$$\|Z_3\|_{l_0} \leq C\alpha^2/3.$$

Thus

$$\|Y' - \alpha d^{-1}\hat{T}\|_{l_0} \leq C\alpha^2,$$

completing the proof for the case $n_0 \leq n \leq 2n_0$.

The general case follows by induction. Assume that the conclusion holds for $n \leq kn_0$, $k \geq 2$. To deal with case $kn_0 < n \leq (k+1)n_0$, let ω_1 be the element of $\mathcal{P}_{n-n_0}$ which contains ω and let $Y_1 := F^{n-n_0}(X|\omega_1)$. Then by induction hypothesis $Y_1 \in \mathcal{H}_C(\alpha)$. Since $Y = F^{n_0}(Y_1|g^{n-n_0}(\omega))$, we obtain $Y \in \mathcal{H}_C(\alpha)$. $\square$

Let us say that a curve $X : [0,1) \to B$ is *admissible* if it is the image of a horizontal curve under F^n for some $n \geq n_0+1$: there exists $\omega \in \mathcal{P}_n$, $x_0 \in B$ such that $X(g^n(\theta)) = f_n(\theta, x_0)$ for all $\theta \in \omega$. By Lemma 7 and Remark on page 5, $X \in \mathcal{H}_C(\alpha)$.

Proposition 1. *There exists $\varepsilon_0 > 0$ and $\kappa > 0$ such that when α is sufficiently small, for any admissible curve $X : [0,1) \to B$, and any $0 \leq \varepsilon < \varepsilon_0$, we have*

$$|\{\theta \in [0,1) : |X(\theta)| < \alpha\varepsilon\}| \leq \varepsilon^\kappa.$$

Proof. By definition, there exists $T \in \mathcal{T}_\phi$ such that $X' - \alpha d^{-1}T$ has C^{l_0}-norm bounded from above by $C\alpha^2$. Then provided that α is small enough, by Lemma 6 for any $\theta \in [0,1)$,

$$\mu\alpha \leq 2\mu\alpha - C\alpha^2 \leq \sum_{i=1}^{l_0} |X^{(i)}(\theta)| \leq \sum_{i=1}^{l_0+1} |X^{(i)}(\theta)| \leq \frac{A}{2}\alpha + C\alpha^2 \leq A\alpha.$$

It follows that we can divide $[0,1)$ as the disjoint union of intervals J_i, $i = 1, 2, \ldots, m$, with the following properties:

- $|J_i| \geq \frac{\mu}{2Al_0}$,
- there exists $j_i \in \{1, 2, \ldots, l_0\}$ such that $|X^{(j_i)}(\theta)| \geq \frac{\mu\alpha}{2l_0}$ for $\theta \in J_i$.

By Lemma 5.3 in [5], $|\{\theta \in J_i : |X(\theta)| < \alpha\varepsilon\}| < 2^{j_i+1}(\frac{2l_0\varepsilon}{\mu})^{\frac{1}{j_i}}$ for all $i \in \{1, 2, \cdots, m\}$ and $\varepsilon > 0$. Take $W = \max\{2^{j_i+1}(\frac{2l_0}{\mu})^{\frac{1}{j_i}} : i = 1, 2, \cdots, l_0\}$. Thus

$$|\{\theta \in [0,1) : X(\theta)| < \alpha\varepsilon\}| < mW\epsilon^{\frac{1}{l_0}} \leq \frac{2Al_0}{\mu}W\varepsilon^{\frac{1}{l_0}} \text{ for all } \varepsilon \in (0,1).$$

Choosing $\kappa = \frac{1}{2l_0}$ and $\varepsilon_0 \in (0,1)$ with $\frac{2Al_0}{\mu}W\varepsilon_0^{\frac{1}{2l_0}} \leq 1$, the proposition follows. $\square$

4. Recurrence to small neighborhoods of the critical circle

We study the recurrence of an admissible curve $X : [0,1) \to B_\alpha := [-2\alpha^{0.6}, 2\alpha^{0.6}]$ to the region $\mathbb{T} \times B_\alpha$. More, precisely, for each $\theta \in [0,1)$, let $n_0(\theta) = 0$ and let

$$n_1(\theta) < n_2(\theta) < \cdots$$

be all the positive integers (finitely or infinitely many) for which there exists θ' which is contained in the same element ω of $\mathcal{P}_{n_k(\theta)}$ as θ, such that $|f_{n_k(\theta)}(\theta', X(\theta'))| \leq \alpha^{0.6}$. Since $F^{n_k(\theta)}(X|\omega)$ is an admissible curve, we have that $f_{n_k(\theta)}(\theta', X(\theta')) \in B_\alpha$ for all $\theta' \in \omega$. Note that if $n_k(\theta) = n$ and ω is the element of $\mathcal{P}_n$ which contains θ, then $n_k(\theta') = n$ for all $\theta' \in \omega$.

For $\alpha > 0$ let $N(\alpha)$ be as in §2.3. The main result of this section is the following:

Proposition 2. *Given $M > 0$, the following holds provided that $\alpha > 0$ is small enough: For each $n \geq 0$ and $k \geq 0$, if ω is an element of $\mathcal{P}_n$ on which $n_k = n$, then*

$$|\{\theta \in \omega : n_{k+1}(\theta) - n_k(\theta) \leq N(\alpha) + M\}| \leq \frac{2}{3}|\omega|.$$

Proposition 2 is a direct consequence of Proposition 3 which will be proved in § 4.1. As a corollary of this result, we shall prove in § 4.2 that suitably truncated vertical derivatives of f_n at $(\theta, X(\theta))$ is exponentially big in n for a.e. θ.

4.1. Recurrence of the critical set.

Proposition 3. *For each $M \in \mathbb{N}$, there exists $\sigma > 0$ such that the following holds provided that $\alpha > 0$ is sufficiently small. Let $X_0 : [0,1) \to B_\alpha$ be an admissible curve and for $n = 0, 1, \ldots,$ let*

$$\Theta_n = \{\theta \in \mathbb{T} : |f_n(\theta, X_0(\theta))| < \sigma\}.$$

Then

$$\left| \bigcup_{n=1}^{N(\alpha)+M} \Theta_n \right| \leq \frac{2}{3}.$$

Proof. Recall that in §2.3, we proved that there exist constants $C_0 \in (0,1)$ and $\xi_0 > 0$ such that the following hold:

- $C_0 \leq |(f^{N(\alpha)})'(c_1)|\alpha \leq C_0^{-1}$,
- for any (θ, x) with $|x| < \sqrt{\alpha}$, $|f_n(\theta, x)| \geq \xi_0$ for all $n = 1, 2, \ldots, N(\alpha)$.

Let $\varepsilon = \varepsilon(M) \in (0,1)$ be a small constant so that for any $x \in [-\varepsilon, \varepsilon]$ and $1 \leq k \leq M$, $|f^k(x)| \geq 2\varepsilon$. Provided that $\alpha > 0$ is small enough, it follows that for any $x \in [-\varepsilon, \varepsilon]$, $\theta \in \mathbb{T}$ and $1 \leq k \leq M$, we have $|f_k(\theta, x)| \geq \varepsilon$.

Let N_0 be a positive integer such that ϕ is not of period d^{-N_0}. Let $N_1 \geq N_0$ be a large integer such that

$$e\frac{A}{C_0} \cdot 4^M |(f^{N_1-1})'(c_1)|^{-1} \leq \frac{\varepsilon}{2}.$$

Let $Y_j(\theta) = f_j(\theta, 0)$, $j = 0, 1, \cdots$. Then

$$Y_{N_1}(\theta) = f^{N_1}(0) + \alpha Q_{N_1}(\theta) + O(\alpha^2),$$

where

$$Q_{N_1}(\theta) = \sum_{k=1}^{N_1} (f^{N_1-k})'(f^k(0))\phi(d^{k-1}\theta).$$

Claim 1. $Q_{N_1}(\theta)$ is not of period d^{-N_0}.

In fact, let

$$\phi(\theta) = c_0 + \sum_{n=1}^{\infty} (c_n e^{2\pi i n\theta} + \overline{c_n} e^{-2\pi i n\theta})$$

be the Fourier series expansion of ϕ. Since ϕ is not of period d^{-N_0}, there exists a minimal positive integer m_0 such that $d^{N_0} \not| m_0$ and $c_{m_0} \neq 0$. Then

$$\int_0^1 Q_{N_1}(\theta) e^{-2\pi i m_0 \theta} d\theta = (f^{N_1-1})'(f(0)) c_{m_0} \neq 0,$$

which implies the claim 1.

For any $\theta \in \mathbb{T}$ let $\theta' = \theta + d^{-N_1} (\mathrm{mod}\ 1)$. Since Q_{N_1} is real analytic and is not of period $1/d^{N_1}$, there exists $\eta = \eta(N_1) > 0$ such that

$$\Omega' := \{\theta \in \mathbb{T} : |Q_{N_1}(\theta) - Q_{N_1}(\theta')| \geq 4\eta\}$$

has Lebesgue measure greater than 0.99. Let Ω be the union of all elements of $\mathcal{P}_{N(\alpha)+M}$ which intersect Ω'. Clearly, $|\Omega| \geq 0.99$. Provided that α is small enough, $N(\alpha) \gg N_1$,

hence the oscillation of $Q_{N_1}(\theta)$ on any element of $\mathcal{P}_{N(\alpha)+M}$ is less than η. Therefore, for each $\theta \in \Omega$,

$$|Q_{N_1}(\theta) - Q_{N_1}(\theta')| \geq 2\eta.$$

It follows that for all $\theta \in \Omega$,

$$\eta\alpha \leq |Y_{N_1}(\theta) - Y_{N_1}(\theta')| \leq A\alpha, \tag{6}$$

provided that α is small enough.

By Lemma 3,

$$\frac{1}{e}|(f^{N(\alpha)-N_1})'(c_{N_1})|\eta\alpha \leq |Y_{N(\alpha)}(\theta) - Y_{N(\alpha)}(\theta')| \leq e|(f^{N(\alpha)-N_1})'(c_{N_1})|A\alpha. \tag{7}$$

For all $n \in [N(\alpha), N(\alpha)+M]$, since $|\frac{\partial f(\theta,x)}{\partial x}| \leq 4$ on I, we have

$$\begin{aligned}
|Y_n(\theta) - Y_n(\theta')| &= |f_{n-N(\alpha)}(g^{N(\alpha)}\theta, Y_{N(\alpha)}(\theta)) - f_{n-N(\alpha)}(g^{N(\alpha)}\theta, Y_{N(\alpha)}(\theta'))| \\
&\leq 4^{n-N(\alpha)}|Y_{N(\alpha)}(\theta) - Y_{N(\alpha)}(\theta')| \\
&\leq eA4^M|(f^{N(\alpha)-N_1})'(c_{N_1})|\alpha \\
&= e\frac{A}{C_0}\cdot 4^M|(f^{N_1-1})'(c_1)|^{-1}\cdot C_0|(f^{N(\alpha)})'(c_1)|\alpha \\
&\leq \varepsilon/2. \qquad (8)
\end{aligned}$$

Now let $\sigma > 0$ be a constant such that

$$\sigma \leq \min\left(\frac{\eta}{4e}\frac{C_0}{|(f^{N_1-1})'(c_1)|}\left(\frac{\varepsilon}{2}\right)^M, \frac{\varepsilon}{4}, \frac{\xi_0}{2}\right).$$

Claim 2. For each $\theta \in \Omega$, θ and θ' does not simultaneously belong to

$$\Theta = \bigcup_{n=1}^{N(\alpha)+M}\{\theta \in [0,1) : |Y_n(\theta)| < 2\sigma\},$$

provided that $\alpha > 0$ is small enough.

To prove this claim, let us assume that $|Y_m(\theta)| < 2\sigma$ for some $m \in \{1, \ldots, N(\alpha)+M\}$. As $|Y_j(\theta)| \geq \xi_0 \geq 2\sigma$ for $1 \leq j \leq N(\alpha)$, we have $m > N(\alpha)$. Since $|Y_m(\theta)| \leq \varepsilon/2$, by the choice of ε, we obtain that for $k \in \{N(\alpha), \ldots, m-1\}$, we have $|Y_k(\theta)| \geq \varepsilon$, hence $[Y_k(\theta), Y_k(\theta')]$ is disjoint from the region $[-\varepsilon/2, \varepsilon/2]$ by (8). Therefore

$$\begin{aligned}
|Y_m(\theta) - Y_m(\theta')| &= |f_{m-N(\alpha)}(g^{N(\alpha)}\theta, Y_{N(\alpha)}(\theta)) - f_{m-N(\alpha)}(g^{N(\alpha)}\theta, Y_{N(\alpha)}(\theta'))| \\
&\geq \left(\frac{\varepsilon}{2}\right)^{m-N(\alpha)}|Y_{N(\alpha)}(\theta) - Y_{N(\alpha)}(\theta')| \geq \left(\frac{\varepsilon}{2}\right)^M\frac{1}{e}|(f^{N(\alpha)-N_1})'(c_{N_1})|\eta\alpha \\
&= \frac{\eta}{e}\frac{C_0}{|(f^{N_1-1})'(c_1)|}\left(\frac{\varepsilon}{2}\right)^M\cdot\frac{|(f^{N(\alpha)})'(c_1)|\alpha}{C_0} \geq 4\sigma,
\end{aligned}$$

hence $|Y_m(\theta')| > 2\sigma$. On the other hand,

$$|Y_m(\theta')| \leq |Y_m(\theta)| + |Y_m(\theta) - Y_m(\theta')| < \varepsilon,$$

which implies that $|Y_k(\theta')| \geq \varepsilon$ for all $1 \leq k \leq N(\alpha)+M$ except for $k = m$. In conclusion, we obtain that $\theta' \notin \Theta$. This finishes the proof of Claim 2.

The claim implies that

$$|\Theta| \leq 0.51.$$

It is clear that there exists $C_2 \geq 1$ such that $|(f^{k_1})'(c_1)| \leq C_2|(f^{k_2})'(c_1)|$ for any $0 \leq k_1 < k_2$. For any $\theta \in \Theta_n$, $n = 1, 2, \ldots, N(\alpha) + M$, we have

$$\begin{aligned}|f_n(\theta, 0) - f_n(\theta, X_0(\theta))| &= |f_{n-1}(g\theta, f_1(\theta, 0)) - f_{n-1}(g\theta, f_1(\theta, X_0(\theta)))| \\ &= |\frac{\partial f_{n-1}(g\theta, \xi)}{\partial x}| \cdot |X_0(\theta)|^2 \text{ for some } \xi \in [f_1(\theta, X_0(\theta)), f_1(\theta, 0)] \\ &\leq \begin{cases} e|(f^{n-1})'(c_1)|\alpha^{1.2}, & n-1 \leq N(\alpha) \\ 4^{n-1-N(\alpha)}e|(f^{N(\alpha)})'(c_1)|\alpha^{1.2}, & n-1 \geq N(\alpha) \end{cases} \\ &\leq C_2 e 4^M |(f^{N(\alpha)})'(c_1)|\alpha^{1.2} \leq \frac{4^M C_2 e}{C_0}\alpha^{0.2} \leq \sigma,\end{aligned}$$

provided that α is small enough. Hence $|f_n(\theta, 0)| < 2\sigma$. This proves that $\bigcup_{n=1}^{N(\alpha)+M} \Theta_n \subset \Theta$, completing the proof of the proposition. $\square$

4.2. **Truncated vertical partial derivative.** Consider an admissible curve $X : [0, 1) \to B_\alpha$. Define $n_k(\theta)$ as above. For each $k = 1, 2, \ldots$ and $M > 0$, define

$$\mathcal{B}_k(M) := \{\theta \in [0, 1) : n_k(\theta) \leq (N(\alpha) + M)k\}.$$

The following is an easy consequence of Proposition 2.

Corollary 1. *Given $M > 0$, the following holds provided that $\alpha > 0$ is small enough. For each $k = 1, 2, \ldots$, we have $|\mathcal{B}_k(M)| \leq 0.8^k$. In particular, for a.e. $\theta \in [0, 1)$, $n_k(\theta) > (N(\alpha) + M)k$ for all k sufficiently large.*

Proof. Let $L > 0$ be a large number such that

$$\sum_{(1-L^{-1})k \leq t \leq k} \binom{k}{t} \left(\frac{2}{3}\right)^t \leq 0.8^k \tag{9}$$

holds for all positive integers k.

For any finite sequence of integers $0 \leq k_1 < k_2 < \cdots < k_t$, let

$$\mathcal{D}(k_1, k_2, \cdots, k_t) = \{\theta \in [0, 1) : n_{k_i+1}(\theta) - n_{k_i}(\theta) \leq N(\alpha) + LM, 1 \leq i \leq t\}.$$

We claim that

$$|\mathcal{D}(k_1, k_2, \cdots, k_t)| \leq (2/3)^t, \tag{10}$$

provided that $\alpha > 0$ is small enough.

To prove the claim, let $\mathcal{F}_k^n$, $k \geq 0$, $n \geq 0$, denote the collection of elements of $\mathcal{P}_n$ on which $n_k(\theta) = n$, and let $\mathcal{F}_k = \bigcup_{n=0}^\infty \mathcal{F}_k^n$. Let D_k denote the union of elements of $\mathcal{F}_k$ which intersects $\mathcal{D}(k_1, k_2, \cdots, k_t)$. Clearly $D_{k+1} \subset D_k$ for all $k \geq 0$. By Proposition 2 (with M replaced by LM), for each $1 \leq i \leq t$, and each element ω of $\mathcal{F}_{k_i}$, we have $|D_{k_i+1} \cap \omega| \leq \frac{2}{3}|\omega|$, provided that $\alpha > 0$ is small enough. Thus $|D_{k_i+1}| \leq (2/3)|D_{k_i}|$. Since $\mathcal{D}(k_1, k_2, \ldots, k_t) \subset D_{k_t+1}$, the inequality (10) follows.

Now let us complete the proof of the corollary. If $\theta \in \mathcal{B}_k(M)$, then

$$\sum_{j=0}^{k-1}(n_{j+1}(\theta) - n_j(\theta) - N(\alpha)) \leq kM.$$

As $n_{j+1}(\theta) - n_j(\theta) \geq N(\alpha)$ holds for all j, it follows that

$$\#\{0 \leq j < k : n_{j+1}(\theta) - n_j(\theta) > LM + N(\alpha)\} \leq \frac{k}{L}.$$

Thus there exist integers $t \geq (1-L^{-1})k$ and $0 \leq k_1 < k_2 < \cdots < k_t$ such that $\theta \in \mathcal{D}(k_1, k_2, \ldots, k_t)$. By (10), $|\mathcal{B}_k(M)|$ is bounded from above by the left hand side of (9). The corollary is proved. $\square$

For $(\theta, x) \in \mathbb{T} \times \mathbb{R}$, and $n \geq 1$, define

$$\frac{\hat{\partial} f_n(\theta, x)}{\partial x} = \prod_{i=0}^{n-1} \max\left(2|f_i(\theta, x)|, 2\alpha\right). \tag{11}$$

Proposition 4. *There exists $\lambda_1 > 1$ such that for each positive integer M, the following holds provided that α is small enough. Let $X : [0,1) \to B_\alpha$ be an admissible curve. Then for a.e. $\theta \in [0,1)$,*

$$\left|\frac{\hat{\partial} f_n(\theta, X(\theta))}{\partial x}\right| \geq \lambda_1^{nM/N(\alpha)}.$$

holds for all n large.

Proof. Let $K > 0$ and $\lambda > 1$ be as in Lemma 2 and let C_0 be as in Lemma 5. Take $M_* \in \mathbb{N}$ with $\lambda^{M_*} K C_0 \geq 1$. Define $n_1, n_2, \ldots$ as above and fix M. We only need to consider those θ for which $n_k(\theta)$ is defined for all $k \geq 0$. By Corollary 1, for a.e. $\theta \in [0,1)$, $n_k(\theta) \geq k(N(\alpha)+M+M_*+1)$ for all k sufficiently large. Let us fix such a θ and prove that the expansion estimate holds. Take $L = L(\theta) > 0$ such that $n_k(\theta) \geq k(N(\alpha)+M+M_*+1)$ for all $k \geq L$. Write $z_i = F^i(\theta, X(\theta))$ and $n_i = n_i(\theta)$. By Lemma 2,

$$\left|\frac{\partial f_{n-n_k-1}(z_{n_k+1})}{\partial x}\right| \geq K\alpha^{0.6}\lambda^{n-n_k-1} \tag{12}$$

for $n \in [n_k, n_{k+1}]$ and

$$\left|\frac{\partial f_{n_{k+1}-n_k-N(\alpha)-1}(z_{n_k+N(\alpha)+1})}{\partial x}\right| \geq K\lambda^{n_{k+1}-n_k-1-N(\alpha)}. \tag{13}$$

By Lemma 5,

$$\left|\frac{\partial f_{N(\alpha)}(z_{n_k+1})}{\partial x}\right| \geq \frac{C_0}{\alpha}. \tag{14}$$

By (13) and (14),

$$\left|\frac{\partial f_{n_{k+1}-n_k-1}(z_{n_k+1})}{\partial x}\right| \geq \frac{C_0 K}{\alpha}\lambda^{n_{k+1}-n_k-1-N(\alpha)}. \tag{15}$$

Thus for $n_k \leq n < n_{k+1}$ with $k \geq L$, using (12) and (15) we have

$$\begin{aligned}
\left|\frac{\hat{\partial} f_n(\theta, X(\theta))}{\partial x}\right| &= \left|\frac{\partial f_n(\theta, X(\theta))}{\partial x}\right| \cdot \prod_{i=0}^{k} \frac{\alpha}{|f_{n_i}(\theta, X(\theta))|} \\
&= \alpha^{k+1}\left|\frac{\partial f_{n-n_k-1}(z_{n_k+1})}{\partial x}\right| \cdot \prod_{i=0}^{k-1}\left|\frac{\partial f_{n_{i+1}-n_i-1}(z_{n_i+1})}{\partial x}\right| \\
&\geq \alpha^{1.6}K^{k+1}C_0^k\lambda^{n-kN(\alpha)-(k+1)} = \alpha^{1.6}K\lambda^{-1}\left(\frac{KC_0}{\lambda^{N(\alpha)+1}}\right)^k \lambda^n \\
&\geq \alpha^{1.6}K\lambda^{-1}\left(\frac{KC_0}{\lambda^{N(\alpha)+1}}\right)^{\frac{n}{N(\alpha)+M+M_*+1}} \lambda^n \quad \text{(when } \alpha \text{ is small enough)} \\
&= \alpha^{1.6}K\lambda^{-1}(KC_0\lambda^{M_*}\lambda^M)^{\frac{n}{N(\alpha)+M+M_*+1}} \geq \alpha^{1.6}K\lambda^{-1} \cdot \lambda^{\frac{nM}{N(\alpha)+M+M_*+1}} \\
&\geq \alpha^{1.6}K\lambda^{-1} \cdot \lambda^{\frac{nM}{2N(\alpha)}} \qquad \text{(when } \alpha \text{ is small enough).}
\end{aligned}$$

Taking $\lambda_1 = \lambda^{\frac{1}{3}}$ gives us the desired estimate. $\square$

5. Exclusion of bad values

We analyze returns of points to the region $\mathbb{T} \times (-\alpha, \alpha)$. Using a large deviation argument from [9], we obtain bounds for such deep returns which, together with the estimate given by Proposition 4, shows that F has a positive vertical Lyapunov exponent almost everywhere.

Write

$$\chi_-(\theta, x) = \liminf_{n \to +\infty} \frac{1}{n} \log \left| \frac{\partial f_n(\theta, x)}{\partial x} \right|.$$

Proposition 5. *Let $X : [0,1) \to B$ be an admissible curve. Then for a.e. $\theta \in [0,1)$, $\chi_-(\theta, X(\theta)) > 0$.*

Proof of the Theorem 1. It suffices to prove that $\chi_-(\theta, x) > 0$ for a.e. $(\theta, x) \in \mathbb{T} \times \mathbb{R}$. Since $|\frac{\partial f_1(\theta,x)}{\partial x}| > \beta > 1$ outside of I, we only need to consider $(\theta, x) \in \mathbb{T} \times B$. By Fubini's theorem, we only need to show that for any $x \in B \setminus \{0\}$, the set

$$Q(x) = \{\theta \in \mathbb{T} : \chi_-(\theta, x) \le 0\}$$

has measure zero. Let n_0 be as in Lemma 1 and let ω be an element of $\mathcal{P}_{n_0+1}$. Then by Lemma 7, $\theta \mapsto f_{n_0+1}((g^{n_0+1}|\omega)^{-1}(\theta), x)$ defines an admissible curve. By Proposition 5, we have

$$\chi_-(F^{n_0+1}(\theta, x)) > 0$$

for a.e. $\theta \in \mathbb{T}$. Clearly, for each $j = 0, 1, \ldots, n_0$, there are only finitely many θ for which $f_j(\theta, x) = 0$. Thus $\chi_-(\theta, x) = \chi_-(F^{n_0+1}(\theta, x)) > 0$ for a.e. $\theta \in \mathbb{T}$. $\square$

Let us turn to the proof of Proposition 5. Without loss of generality, let us consider an admissible curve $X : [0,1) \to B_\alpha$, where $B_\alpha = [-2\alpha^{0.6}, 2\alpha^{0.6}]$ as before. Let

$$Q := \{\theta \in [0,1) : f_i(\theta, X(\theta)) = 0 \text{ for some } i = 0, 1, \ldots\},$$

which is a countable set. As in § 4, we define $0 = n_0(\theta) < n_1(\theta) < n_2(\theta) < \cdots$.

Let $\varepsilon_0 > 0$ be as in Proposition 1. Let $\varepsilon_1 \in (0, \varepsilon_0)$ be a small constant and let

$$\Delta = \Delta(\varepsilon_1) := \log_d \frac{1}{\varepsilon_1}. \tag{16}$$

For any $\theta \in [0,1) \setminus Q$ and $k \ge 0$, if $n_k(\theta)$ is defined, then we define $\hat{q}_k(\theta)$ to be the integer such that

$$d^{-\hat{q}_k(\theta)+1}\alpha > |f_{n_k(\theta)}(\theta, X(\theta))| \ge d^{-\hat{q}_k(\theta)}\alpha.$$

Moreover, define $q_k(\theta) = 0$ if $d^{-(\hat{q}_k(\theta)-1)} \ge \varepsilon_1$ and $q_k(\theta) = \hat{q}_k(\theta)$ otherwise. If $n_k(\theta)$ is not defined, then we set $q_k(\theta) = 0$.

Therefore $q_k(\theta) > 0$ implies that

$$q_k(\theta) > \Delta. \tag{17}$$

For $c > 0$ and $K \ge 1$, let

$$E_K(c) := \{\theta \in [0,1) \setminus Q : \sum_{k=0}^{K} q_k(\theta) \ge cK\}.$$

Lemma 8. *For any $c>0$, the following holds for all K large:*

$$|E_K(c)| \le d^{-\frac{\kappa}{6}c\sqrt{K}},$$

provided that ε_1 was chosen small enough, where κ is as in Proposition 1.

Proof. We decompose the set

$$E_K(c) = E_K^1(c) \cup E_K^2(c),$$

where

$$E_K^2(c) := \{\theta \in E_K(c) : \max_{0\le k\le K} q_k(\theta) \ge [\sqrt{K}]\},$$
$$E_K^1(c) := E_K(c) \setminus E_K^2(c).$$

By Proposition 1,

$$|E_K^2(c)| \le \sum_{k=0}^{K} |\{\theta \in E_K(c) : q_k(\theta) \ge [\sqrt{K}]\}| \le (K+1)d^{-\kappa([\sqrt{K}]-1)}. \tag{18}$$

Let us estimate $|E_K^1(c)|$. To this end, let $m = [\sqrt{K}]$ and let

$$\mathcal{A}_p = \{0 \le k \le K : k \equiv p \mod m\}.$$

Let

$$E_{K,p}^1(c) := \left\{\theta \in E_K^1(c) : \sum_{k\in\mathcal{A}_p} q_k(\theta) \ge \frac{cK}{m}\right\}.$$

Then

$$E_K^1(c) = \bigcup_{p=0}^{m-1} E_{K,p}^1(c).$$

Let us now fix p, and estimate $|E_{K,p}^1(c)|$. For each $\mathbf{r} = (r_k)_{k\in\mathcal{A}_p} \in \{0,1,\ldots,[\sqrt{K}]\}^{\mathcal{A}_p}$, let $\|\mathbf{r}\| = \sum_k r_k$, and let

$$E_{K,p}^1(c,\mathbf{r}) = \{\theta \in E_{K,p}^1(c) : q_k(\theta) = r_k \text{ for all } k \in \mathcal{A}_p\}.$$

Claim. For each $\mathbf{r} \in \{0,1,\ldots,[\sqrt{K}]\}^{\mathcal{A}_p}$, we have

$$|E_{K,p}^1(c,\mathbf{r})| \le d^{-\frac{\kappa}{2}\|\mathbf{r}\|} \tag{19}$$

Proof of Claim. Let $k_1 < k_2 < \ldots < k_\nu$ be all the elements in $\mathcal{A}_p$ such that $r_{k_j} > 0$.

For each $\theta \in E_{K,p}^1(c,\mathbf{r})$, let $\omega_j'(\theta)$ (resp. $\omega_j(\theta)$) be the element of $\mathcal{P}_{n_{k_j}(\theta)}$ (resp. $\mathcal{P}_{n_{k_j}(\theta)+r_{k_j}}$) which contains θ. Let

$$\Omega_j' := \bigcup_{\theta\in E_{K,p}^1(c,\mathbf{r})} \omega_j'(\theta), \quad \Omega_j := \bigcup_{\theta\in E_{K,p}^1(c,\mathbf{r})} \omega_j(\theta).$$

Since

$$n_{k_{j+1}}(\theta) - n_{k_j}(\theta) - r_{k_j} \ge mN(\alpha) - r_{k_j} \ge 0,$$

holds for all θ, $\Omega_{j+1}' \subset \Omega_j$.

For each $\omega_j'(\theta)$, since $F^{n_{k_j}}(X|\omega_j')$ is an admissible curve, we have that

$$|f_{n_{k_j}}(\theta', X(\theta'))| \le |f_{n_{k_j}}(\theta, X(\theta))| + A\alpha d^{-r_{k_j}} \le (d+A)\alpha d^{-r_{k_j}},$$

holds for all $\theta' \in \omega'_j(\theta) \cap \Omega_j$. By Proposition 1, it follows that

$$\frac{|\omega'_j(\theta) \cap \Omega_j|}{|\omega'_j(\theta)|} \leq (d+A)^\kappa d^{-\kappa r_{k_j}} \leq d^{-\kappa r_{k_j}/2},$$

provided that we have chosen ε_1 small enough in (16) so that $r_{k_j} \geq \Delta$ is large.

Thus for each j,

$$|\Omega_j| \leq d^{-\kappa r_{k_j}/2} |\Omega'_j|,$$

hence

$$|\Omega_{j+1}| \leq d^{-\kappa r_{k_{j+1}/2}} |\Omega_j|.$$

It follows that

$$|E^1_{K,p}(c, \mathbf{r})| \leq |\Omega_\nu| \leq d^{-\frac{\kappa}{2} \sum\limits_{k \in \mathcal{A}_p} r_k}.$$

□

To complete the proof of the lemma, let us estimate the number $Q(K,p,R)$ of $\mathbf{r} = (r_k) \in \{0, 1, \ldots, [\sqrt{K}]\}^{\mathcal{A}_p}$ with $\sum_{k \in \mathcal{A}_p} r_k = R$ and with $E^1_{K,p}(c, \mathbf{r}) \neq \emptyset$. Clearly,

$$Q(K,p,R) \leq \sum_{\nu=1}^{\min(\nu_p, [R/\Delta])} \begin{pmatrix} \nu_p \\ \nu \end{pmatrix} \begin{pmatrix} R - \nu[\Delta] \\ \nu - 1 \end{pmatrix},$$

where $\nu_p = \#\mathcal{A}_p \approx \sqrt{K}$. The first binomial coefficient corresponds to the number of possible positions for which $r_k \neq 0$, and the second corresponds to the distribution of the sum into terms. Assuming ε_1 small so that Δ large, let us prove that

$$Q(K,p,R) \leq d^{\kappa R/4} \tag{20}$$

holds for all K sufficiently large and $R \geq cK/m$.

To this end, by Stirling's formula we first observe that for each $1 \leq \nu \leq [R/\Delta]$,

$$\begin{pmatrix} R - \nu[\Delta] \\ \nu - 1 \end{pmatrix} \leq \begin{pmatrix} R \\ \nu \end{pmatrix} \leq \begin{pmatrix} R \\ [R/\Delta] \end{pmatrix} \leq d^{\kappa R/8}$$

for R sufficiently enough, provided that ε_1 small . Thus

$$Q(K,p,R) \leq \sum_{\nu=1}^{\nu_p} \begin{pmatrix} \nu_p \\ \nu \end{pmatrix} d^{\kappa R/8} \leq 2^{\nu_p} d^{\kappa R/8}.$$

In particular, there is a constant C such that (20) holds if $R > C\sqrt{K}$. Assume that $R \leq C\sqrt{K}$. Provided that Δ is large enough, $[R/\Delta] < \nu_p$, so

$$Q(K,p,R) \leq d^{\kappa R/8} \sum_{\nu=1}^{[R/\Delta]} \begin{pmatrix} \nu_p \\ \nu \end{pmatrix} \leq d^{\kappa R/8} [R/\Delta] \begin{pmatrix} \nu_p \\ [R/\Delta] \end{pmatrix}.$$

Since $\nu_p \approx \sqrt{K}$ and $R \geq cK/m \approx c\sqrt{K}$, (20) follows.

By the Claim, we obtain

$$\begin{aligned} |E^1_{K,p}(c)| &= \sum_{\mathbf{r} \in \{0,1,\ldots,[\sqrt{K}]\}^{\mathcal{A}_p} : \|\mathbf{r}\| \geq \frac{cK}{m}} |E^1_{K,p}(c, \mathbf{r})| \leq \sum_{\mathbf{r} \in \{0,1,\ldots,[\sqrt{K}]\}^{\mathcal{A}_p} : \|\mathbf{r}\| \geq \frac{cK}{m}} d^{-\frac{\kappa}{2}\|\mathbf{r}\|} \\ &\leq \sum_{R \geq cK/m} Q(K,p,R) d^{-\frac{\kappa}{2}R} \leq \sum_{R \geq cK/m} d^{-\kappa R/4} \leq \frac{1}{1 - d^{-\frac{\kappa}{4}}} d^{-\kappa c\sqrt{K}/4}. \end{aligned}$$

It follows that

$$|E_K^1(c)| \le \sum_{p=0}^{m-1} |E_{K,p}^1(c)| \le m\frac{1}{1-d^{-\frac{\kappa}{4}}} d^{-\kappa c\sqrt{K}/4} \le d^{-\kappa c\sqrt{K}/5} \tag{21}$$

when K large. Combining (18) and (21) completes the proof of the lemma. □

Proof of Proposition 5. By Lemma 8, fix a small constant $\varepsilon_1 > 0$ so that $\sum_{K=1}^{\infty} |E_K(1)|$ converges. Hence, for a.e. $\theta \in [0,1) \setminus Q$, there exists $K_0 = K_0(\theta)$ such that $\theta \notin E_K(1)$ for all $K \ge K_0$.

Choose a large positive integer M such that $\lambda' = \lambda_1^M \varepsilon_1 d^{-2} > 1$, where $\lambda_1 > 1$ is as in Proposition 4. By Proposition 4, provided that $\alpha > 0$ is small enough, the following holds: for a.e. $\theta \in [0,1)$,

$$\frac{\hat{\partial} f_n(\theta, X(\theta))}{\partial x} \ge \lambda_1^{\frac{Mn}{N(\alpha)}}$$

holds for all large n.

Now take θ with the above properties. For any $n \ge 1$, and let k_n be maximal such that $n_{k_n}(\theta) \le n$. Then $k_n \le n/N(\alpha)$ and for all large n,

$$\begin{aligned}\left|\frac{\partial f_n(\theta, X(\theta))}{\partial x}\right| &= \frac{\hat{\partial} f_n(\theta, X(\theta))}{\partial x} \prod_{k=0}^{k_n} \frac{|f_{n_k}(\theta, X(\theta))|}{\alpha} \\ &\ge \frac{\hat{\partial} f_n(\theta, X(\theta))}{\partial x} \prod_{k=0}^{k_n} \min\left(\frac{\varepsilon_1}{d}, d^{-q_k(\theta)}\right) \\ &\ge \lambda_1^{nM/N(\alpha)} \left(\frac{\varepsilon_1}{d}\right)^{k_n+1} d^{-\sum_{k=0}^{k_n} q_k(\theta)}.\end{aligned}$$

We may assume that $k_n \to \infty$, for otherwise, $\chi_-(\theta, X(\theta)) > 0$ obviously holds. For n large, $k_n \ge K_0$, hence

$$\sum_{k=0}^{k_n} q_k(\theta) \le k_n.$$

It follows that for n large,

$$\left|\frac{\partial f_n(\theta, X(\theta))}{\partial x}\right| \ge \lambda_1^{nM/N(\alpha)} \varepsilon_1^{k_n+1} d^{-(2k_n+1)} \ge \frac{\varepsilon_1}{d}\left(\lambda_1^M \varepsilon_1 d^{-2}\right)^{n/N(\alpha)} = \frac{\varepsilon_1}{d}(\lambda')^{n/N(\alpha)}.$$

Therefore $\chi_-(\theta, X(\theta)) > 0$. □

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